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Semantic Incompleteness of Quantum Physics

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We discuss the completeness of quantum physics (QP) from a nonrealistic viewpoint. To this end we make use of the formalized language L for QP that we introduced in a recent paper and show that QP is "incomplete" both in an intuitive sense and in a more formal logical sense. We also show that a pure state is not physically equivalent to the physical property which characterizes it in QP, and that the set of all properties whose truth value can be predicted for a physical object in the state S coincides with the set of all properties which are "certainly true" or "certainly false" in S. These results lead us to introduce a notion of "compatibility" between states which can be applied to the EPR experiment, in order to prove that no quantum paradox follows from it if our interpretation of states and physical properties is accepted.

1. INTRODUCTION

In their famous paper, "Can Quantum Mechanical Description of Reality Be Considered Complete?" Einstein, Podolski, and Rosen (1935) (EPR) charged quantum physics (QP) with being "incomplete" in a sense which goes beyond the usual meaning of this word in logic and epistemology; to be precise, they argued from the analysis of a suitable thought-experiment that an "element of physical reality" could be simultaneously attributed to some physical "quantities" which nevertheless could not be simultaneously evaluated by making use of the laws of QP.

It should be noted that the realistic attitude of EPR in the above paper can be considered "weak" (in a sense that will be clarified in Section 3). But it is well known that the EPR argument has stimulated the production of a huge amount of literature on the subject; in this context, many physicists adopted a "strong" realistic attitude and interpreted the EPR reasoning as the discovery of a "paradox" in QP (Selleri, 1988), or deduced "paradoxes" by further elaborating the EPR argument.

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If analyzed from a semantic viewpoint, "strong" realism requires that every physical theory can be completely interpreted, the interpretation not being considered a model for the theory (Braithwaite, 1953) but, rather, a picture of some ontologically existing underlying reality. From this point of view, this philosophical attitude contrasts with a more analytical and nonrealistic position, according to which the language of any physical theory necessarily contains theoretical terms which can be interpreted on a model (or "intended interpretation") for making reasoning easier, but which do not necessarily refer to ontologically existing entities; indeed, their physical meaning is established by means of "correspondence rules" which associate derived theoretical terms with observative terms, i.e., terms interpreted on physical quantities (Hempel, 1965; Carnap, 1966).

The problem of the completeness of a physical theory $\mathscr{T}$, or the problem of its "paradoxes,"[2] do not disappear in a nonrealistic framework, but receive a different formulation; in particular, the former becomes a problem of "semantic" rather than "ontological" completeness, and a thorough discussion of it would require an adequate formalization of the language of $\mathscr{T}$. This could be done (see Section 3) by constructing a predicate calculus L^* where quantification of predicative variables be admitted, so that the general physical laws of $\mathscr{T}$ can be stated by means of L^*. By making reference to this language L^*, the completeness problem can be posed in more formal terms, and "weak" and "strong" realism can be adequately characterized.

We are interested here in classical physics (CP) and QP. The construction of L^* for these theories is a major task, and we cannot tackle it now. This notwithstanding, we can attain some relevant results regarding the completeness problem by making use of the formal language L that we have proposed in a recent paper (Garola, 1991; briefly, G.91 in the following). Indeed, L does not completely formalize the language of CP and QP, but it can be seen as an "observative part" of the broader language L^* that would be required in order to formalize these theories (i.e., L is a sublanguage of L^* which is interpreted over the empirical domain of CP and QP). Thus, we can pose the completeness problem with reference to L rather than to L^*, and it is apparent that a negative answer in the former case would imply a negative answer in the latter; because of this, we shall see that L is sufficient for our purposes in this paper.

Let us briefly summarize our main results. We discuss in Section 2 the "incompleteness" of QP from an intuitive viewpoint, by making use of our

[2]The term "*paradox*" is used here in order to indicate a result in the theory which is heavily counterintuitive or, more rigorously, which contradicts some epistemological requirements regarding the theory; a paradox must then be distinguished from an "antinomy," which is an internal contradiction in the theory.

language L and of standard quantum mechanical notions; we also show that physical states can be considered physically equivalent to testable physical properties in CP but not in QP (this result is consistent with the standard conception of states as "amounts of information" in QP), though every state can be characterized by a suitable physical property both in CP and in QP. Then we show in Section 3, by making use of the reformulation of the basic axioms of QP provided in G.91, that QP is semantically incomplete with respect to L in a standard sense in logic, while CP is semantically complete. Based on this result, we introduce a (binary) "compatibility" relation on the set of all states, which is used in Section 4 in order to show that no quantum paradox follows from the EPR experiment in our framework (it is interesting to note that EPR deduced the ontological incompleteness of QP by reasoning about their "thought-experiment," while we prove the semantic incompleteness of QP by means of general logical tools, then make use of this result in order to disprove the existence of paradoxes following from the EPR experiment).

2. STATES AND PHYSICAL PROPERTIES

As we have anticipated in the Introduction, in this section we intend to provide an intuitive treatment of the incompleteness of QP by making use of the formalized language that we have proposed in G.91. Our treatment will also put in evidence some important features of the concept of state in QP.

Let us recall the essentials of our approach in G.91. Our classical extended language L is a *predicate calculus* of the first order with:

(i) A set X of *individual variables* $x, y, \ldots$.
(ii) A set $\mathscr{P}$ of *monadic predicates.*
(iii) *Standard connectives* $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$.
(iv) *Standard quantifiers* $\exists, \forall$.
(v) A family $(\pi_r)_{r\in[0,1]}$ of *"statistical quantifiers."*

The predicates in L are divided into two classes, the set $\mathscr{S}$ of "symbols of states" and the set $\mathscr{E}$ of "symbols of effects." By adopting Ludwig's analysis of experimental apparatuses (Ludwig, 1983), with some differences that are epistemologically and mathematically important, but that will not be recalled here for brevity's sake, every symbol of state $S\in\mathscr{S}$ is interpreted, intensionally, as a class $[p_S]$ of physically equivalent preparing devices, or "state"; analogously, every symbol of effect $E\in\mathscr{E}$ is interpreted, intensionally, as a class $[e_E]$ of physically equivalent dichotomic (yes–no) registering devices, or "effect" (by abuse of language we will not distinguish between "symbol of state" and "state" or "symbol of effect" and "effect" in the following).

The extensions of these predicates are defined by making reference to the concept of "laboratory," or space-time domain in the actual world; we denote by I the set of all laboratories, and by $\tilde{I}$ a suitable subset of laboratories which are "statistically relevant," i.e., that contain a "large" number of physical objects (see G.91, Sections 1.7 and 1.8). Thus, for every $i \in I$, the extension $\rho_i(S)$ of the state $S \in \mathcal{S}$ is a set which is physically interpreted as the set of all individual physical systems, or "physical objects," prepared in i by devices belonging to $[p_s]$; similarly, the extension $\rho_i(E)$ of the effect $E \in \mathcal{E}$ is a set which is physically interpreted as the set of all physical objects in i which would yield the answer yes if tested by means of a device belonging to $[e_E]$. Consequently, a set D_i is associated with every $i \in I$ that we call the "domain" of i and is interpreted as the set of all physical objects that are prepared in i by means of preparing devices; furthermore, an interpretation σ of the (individual) variables of L is defined as a mapping

$$\sigma: \quad (i, x) \in I \times X \rightarrow \sigma_i(x) \in D_i$$

which, for every $i \in I$, maps the variables in L into elements of D_i (by abuse of language we call "physical objects" these elements, or the variables themselves whenever an interpretation σ is understood).

Consistently with the above interpretation of L, it can be assumed that, for every $i \in I$, the set of all states induces a partition of D_i [i.e., for every S, $S' \in \mathcal{S}$, $\rho_i(S) \cap \rho_i(S') = \varnothing$ whenever $S \neq S'$, and $\bigcup_{S \in \mathcal{S}} \rho_i(S) = D_i$]. On the contrary, for every E, $E' \in \mathcal{E}$, $E \neq E'$ does not imply that $\rho_i(E) \cap \rho_i(E') = \varnothing$, and the set $\mathcal{E}$ can be partially ordered by the relation $<$ defined as follows:

$$\text{for every } E, E' \in \mathcal{E}, \quad E < E' \quad \text{iff} \quad \text{for every } i \in \tilde{I}, \quad \rho_i(E) \subseteq \rho_i(E')$$

We have proved in G.91 that the poset $(\mathcal{E}, <)$ contains a proper subposet $(\mathcal{E}_E, <)$, the poset of all "symbols of exact effects" (briefly, "exact effects"), which is a lattice and is isomorphic, under reasonable physical assumptions, to Mackey's (1963) lattice of questions, or Piron's (1976) lattice of propositions; hence, it is a complete, orthocomplemented, atomic lattice, which is distributive in CP, weakly modular, and satisfying the covering law in QP. It follows that all elements in $\mathcal{E}_E$ can be associated with (testable) physical properties, and by abuse of language we simply call them "properties" in the rest of this paper.

Let us come to the (Tarskian) concept of truth in our language. We say that the atomic wff $S(x)$ is true in the laboratory i for a given interpretation σ of the variables iff $\sigma_i(x) \in \rho_i(S)$. Analogously, we say that the atomic formula $E(x)$ is true in a laboratory i for a given interpretation σ of the variables iff $\sigma_i(x) \in \rho_i(E)$. Because of our general interpretation of L, it follows that $S(x)$ is true (false) in i iff the physical object $\sigma_i(x)$ has (has not) been prepared in i by means of some preparation procedure belonging to

$[p_s]$ (hence, for every $S \in \mathscr{S}$ the truth value of the wff $S(x)$ can be considered as actually known in our language whenever the interpretation σ is given); we briefly say in this case that "x is (is not) in the state S" in the laboratory i, leaving implicit the reference to the interpretation σ. Analogously, $E(x)$ is true (false) in i iff the physical object $\sigma_i(x)$ would yield the answer yes (no) if it should be tested by means of a registering device belonging to $[e_E]$ (because of this interpretation, for every $E \in \mathscr{E}$ the truth value of the wff $E(x)$ can be considered assigned but not necessarily known in our language whenever the interpretation σ is given). Let $E \in \mathscr{E}_E \subset \mathscr{E}$ ($\subset$ denotes strict inclusion in this paper), and let σ be such that $E(x)$ is true (false) in i; we briefly say that "x has (has not) the property E in the laboratory i" or, equivalently, that "E is true (false) in i for the physical object x," again leaving implicit the reference to the interpretation σ. Finally, a truth value is assigned to all molecular and quantified wffs of L by making use of standard definitions in classical logic, suitably extended so that truth values can also be attributed to wffs containing statistical quantifiers.

Let us consider now the completeness problem. We recall that we have discussed in G.91 the "breakdown of strict determinism" in QP. This breakdown can be expressed in the present framework by saying (in absence of superselection rules) that for every state S (respectively, property E) there is at least one property E (respectively, state S) such that, for every $i \in \tilde{I}$,

$$\varnothing \neq \rho_i(S) \cap \rho_i(E) \neq \rho_i(S)$$

i.e., some physical objects in the state S have the property E, some have not (an example is provided in the Hilbert space model for QP by the property E represented by a projection P_E and by a pure state S represented by a vector which neither belongs to the kernel nor to the range of P_E). It follows that QP is "incomplete" in the intuitive sense that the knowledge of the state of a physical object x does not allow us to predict all physical properties of x. This result may seem trivial, but it must be stressed that in our context it makes sense to assign a truth value to every physical property whenever the physical object x is specified, in contrast with the canonical interpretation of QP provided by the Copenhagen school.

Let us discuss the above argument in more detail.

First, let us note that, by making use of the above definitions, a "true domain" $\mathscr{E}_{ix}^{\mathrm{T}}$ and a "false domain" $\mathscr{E}_{ix}^{\mathrm{F}}$ can be associated, whenever an interpretation σ of the variables is given, to every physical object $x \in X$ in every laboratory $i \in I$ by means of the following definitions:

$$\mathscr{E}_{ix}^{\mathrm{T}} = \{E \in \mathscr{E}_E \mid E \text{ is true in } i \text{ for the physical object } x\}$$

$$\mathscr{E}_{ix}^{\mathrm{F}} = \{E \in \mathscr{E}_E \mid E \text{ is false in } i \text{ for the physical object } x\}$$

Since we adopted a Tarskian truth theory, $\mathscr{E}^{\mathrm{T}}_{ix} \cap \mathscr{E}^{\mathrm{F}}_{ix} = \varnothing$; furthermore, $\mathscr{E}^{\mathrm{T}}_{ix} \cup \mathscr{E}^{\mathrm{F}}_{ix} = \mathscr{E}_E$, since the truth value of every wff of the form $E(x)$ is assigned. Physically, these equalities mean that, whenever the physical object x is specified in a laboratory i, all physical properties are either true or false for x.

Second, let us introduce, for every state $S \in \mathscr{S}$, the set $\mathscr{E}_S \subset \mathscr{E}_E$ of all exact effects such that, for every $i \in \tilde{I}$ and $E \in \mathscr{E}_S$, $\rho_i(S) \subseteq \rho_i(E)$, and call E_S the greatest lower bound of $\mathscr{E}_S$. It follows that $\mathscr{E}_S$ is the set of all properties that are "certainly true" in the state S, i.e., of all properties $E \in \mathscr{E}_E$ such that, for every laboratory $i \in \tilde{I}$, "$S(x)$ true in i" implies "$E(x)$ true in i," whatever the interpretation of the variables may be. In addition, it can be proved that $E_S \in \mathscr{E}_S$ because of quantum mechanical laws. Thus we can write

$$\begin{aligned}\mathscr{E}_S &= \{E \in \mathscr{E}_E \mid \text{for every } i \in \tilde{I},\ \rho_i(S) \subseteq \rho_i(E)\}\\ &= \{E \in \mathscr{E}_E \mid \text{for every } i \in \tilde{I},\ (\forall x)(S(x) \rightarrow E(x)) \text{ is true in } i\}\\ &= \{E \in \mathscr{E}_E \mid E_S < E\}\end{aligned}$$

Furthermore, for every $E \in \mathscr{E}_E$, let us denote by $E^{\perp}$ the orthocomplement of E in the orthocomplemented lattice $(\mathscr{E}_E, <)$ [we have shown in G.91 that $E^{\perp}$ is the unique exact effect such that, for every $i \in \tilde{I}$, $\rho_i(E^{\perp}) = D_i \setminus \rho_i(E)$] and for every state $S \in \mathscr{S}$ let us introduce the set $\mathscr{E}^{\perp}_S \subset \mathscr{E}_E$ of all effects that are orthocomplement of elements in $\mathscr{E}_S$. Then the set $\mathscr{E}^{\perp}_S$ can be interpreted as the set of all properties that are "certainly false" in the state S, i.e., of all properties $E \in \mathscr{E}_E$ such that, for every laboratory $i \in \tilde{I}$, "$S(x)$ true in i" implies "$E(x)$ false in i," whatever the interpretation of the variables may be.[3] By making use of some results obtained in G.91, we easily get

$$\begin{aligned}\mathscr{E}^{\perp}_S &= \{E \in \mathscr{E}_E \mid E = E_0^{\perp},\ E_0 \in \mathscr{E}_S\}\\ &= \{E \in \mathscr{E}_E \mid \text{for every } i \in \tilde{I},\ \rho_i(S) \cap \rho_i(E) = \varnothing\}\\ &= \{E \in \mathscr{E}_E \mid \text{for every } i \in \tilde{I},\ \rho_i(E_S) \cap \rho_i(E) = \varnothing\}\\ &= \{E \in \mathscr{E}_E \mid \text{for every } i \in \tilde{I},\ (\forall x)(S(x) \rightarrow \neg E(x)) \text{ is true in } i\}\\ &= \{E \in \mathscr{E}_E \mid E < E^{\perp}_S\}\end{aligned}$$

[3]We define the terms "*certainly true*" and "*certainly false*" here bearing in mind the truth value "certain," or "true," introduced in the Piron (1976) approach to QP; but these terms denote metalinguistic properties of symbols of L and must not be considered truth values in our approach.

It follows from the above definitions and properties of $\mathscr{E}_S$ and $\mathscr{E}_S^\perp$ that, whenever an interpretation σ of the variables is given, the following (metalinguistic) implication holds for every $i \in \tilde{I}$, $x \in X$, $S \in \mathscr{S}$:

$$S(x) \text{ is true in } i \qquad \textbf{implies} \qquad \mathscr{E}_S \subseteq \mathscr{E}_{ix}^{\mathrm{T}}, \mathscr{E}_S^\perp \subseteq \mathscr{E}_{ix}^{\mathrm{F}}$$

Now, the equality sign holds in CP (see G.91), while strict inclusion holds in QP because of the aforesaid breakdown of strict causality. Hence, in particular, $\mathscr{E}_{ix}^{\mathrm{T}}$, $\mathscr{E}_{ix}^{\mathrm{F}}$ may change in QP whenever the interpretation σ is changed in such a way that $S(x)$ remains true. This implies that QP is "incomplete," in the sense specified above.

Our main goals in this section are thus achieved. We adjoin that we call the "certainty domain" D_S of the state $S \in \mathscr{S}$ in the following the set $D_S = \mathscr{E}_S \cup \mathscr{E}_S^\perp$, which collects all properties whose truth values are certain in every laboratory $i \in \tilde{I}$ whenever the physical object which is being considered is in the state S (hence $D_S = \mathscr{E}_E$ in CP, while $D_S \subset \mathscr{E}_E$ in QP), and conclude with the following remark on states and properties, which is important whenever the problem of the "paradoxes" in QP is discussed.

Remark 2.1. Whenever S belongs to the set $\mathscr{S}_P$ of pure states, it can be shown (see G.91; for brevity, we do not report here the definition of pure state in the framework of our approach) that S is the unique state such that, for every $i \in \tilde{I}$, $\rho_i(S) \subseteq \rho_i(E_S)$. Therefore the "support" E_S of S characterizes S if $S \in \mathscr{S}_P$. However, this characterization must be carefully pondered; in fact it must be clearly understood that S and E_S do not necessarily have the same extension in every laboratory $i \in \tilde{I}$, and that the links between $\rho_i(S)$ and $\rho_i(E_S)$ are different in CP and in QP.

Let us consider CP. Here, a "classical physics condition" holds from which the following statements can be deduced (see G.91).

Proposition 2.2. For every $i \in \tilde{I}$, $S \in \mathscr{S}_P$, $E \in \mathscr{E}_E$, either $\rho_i(S) \subseteq \rho_i(E)$ or $\rho_i(S) \cap \rho_i(E) = \varnothing$.

Proposition 2.3. For every $i \in \tilde{I}$, $S \in \mathscr{S}_P$, $\rho_i(S) = \rho_i(E_S)$.

Proposition 2.2 is intuitively correct in CP; indeed, it means that in every laboratory $i \in \tilde{I}$ all physical objects which are in the pure state S either have the property E or they do not. Proposition 2.3 means that, for every pure state S and laboratory $i \in \tilde{I}$, E_S is true for a given physical object x iff x is in the state S, so that S and E_S are in some sense interchangeable, and every pure state can be identified with a suitable physical property.

Let us consider QP. Here neither Proposition 2.2 nor Proposition 2.3 hold because of the breakdown of strict determinism in this theory; instead we get that, for every pure state S and laboratory $i \in \tilde{I}$, $\rho_i(S) \subset \rho_i(E_S)$. Hence, let $S \in \mathscr{S}_P$, $i \in \tilde{I}$; then "$S(x)$ is true in i" [i.e., an interpretation σ of the variables is given such that $\sigma_i(x) \in \rho_i(S)$] implies "$E_S(x)$ is true in i" [i.e., $\sigma_i(x) \in \rho_i(E_S)$] because of the definition of E_S, but the converse implication does not hold. Physically this means that a positive answer to the test whether the physical object x has the property E_S does not imply that x is in the state S. Thus, even if E_S characterizes S, from a physical viewpoint the state S and the property E_S cannot be identified in QP.

3. THE SEMANTIC INCOMPLETENESS OF QP

We mainly intend to prove in the present section that QP is semantically incomplete in a technical sense in logic, which formalizes the intuitive notion of incompleteness introduced in Section 2. To this end we will make use, in particular, of the reformulation of the basic axioms of QP provided in G.91.

Let us begin with a brief analysis of the basic epistemological distinction between wffs that can be endowed with a truth value, testable wffs, and deducible (in the framework of some physical theory) wffs.

Let L^* be a formal language, let ψ be the set of all wffs of L^*, and let $\mathcal{T}$ be a physical theory stated in L^* (i.e., all axioms and all sets of specific assumptions of $\mathcal{T}$ are wffs of L^*). Then two kinds of semantic interpretations can be provided which are epistemologically different. The "intended physical interpretation" yields a physical model of the formal structure, being a complete and direct interpretation of the latter over a domain of (theoretical) physical entities. The "correspondence rules" provide a partial and indirect empirical interpretation of the formal structure over a domain of observative entities (Braithwaite, 1953; Carnap, 1966) (to be precise, the correspondence rules provide an interpretation of a part of L^*, called the "theoretical language" of $\mathcal{T}$, onto another part of L^*, called the "observative language" of $\mathcal{T}$, which is endowed with a complete and direct interpretation, in the sense of Tarski, on a domain of observative entities). In both cases these semantic interpretations are constructed following some general rules (among these, explicitly or not, a model-theoretic semantics, which yields a complete interpretation of the logical symbols in L^* and defines the concept of logical truth); they determine, in particular, the subset ψ_S of all wffs in L^* which are endowed with a truth value (true or false). In the former case (intended interpretation) ψ_S coincides with the set ψ_C of all closed wffs of ψ (i.e., the set of all wffs where variables either do not appear or are quantified); indeed the interpretation is assumed to be complete (and direct). In the latter case (interpretation via correspondence rules) ψ_S is the join of the set ψ_A of all atomic closed wffs that are interpreted via correspondence rules and the set of all molecular and quantified wffs that can be obtained, via formation rules, from atomic formulas in ψ_A, hence $\psi_S \subseteq \psi_C$.

Let us come now to the subset of all testable wffs. We have already observed elsewhere (Garola, 1991*b*) that a physical theory determines a subset of formulas in its language L^* which are considered empirically testable; we have also underlined the epistemological relevance of distinguishing the metalinguistic concept of truth from the concept of testability (Garola, 1992). Thus we introduce the set ψ_T of all testable wffs, which generally depends on $\mathscr{T}$, must contain ψ_A, and is a subset of the set ψ_S of all wffs that can be endowed with a truth value [indeed a truth value is presupposed in the very act of testing; see Popper (1969)], but does not necessarily coincide with it, so that $\psi_T \subseteq \psi_S$.[4]

Finally, let $\mathscr{A}$ be a set of specific assumptions (which can be empty), and let us consider the set ψ_D of all wffs of L^* which are theorems following from the axioms of $\mathscr{T}$ and from the wffs in $\mathscr{A}$; we call ψ_D a set of $\mathscr{T}$-deducible wffs (briefly, "deducible" wffs whenever the theory $\mathscr{T}$ is understood); furthermore, we call $\psi_D^\perp$ the set of all the negations of wffs of ψ_D and put $\psi_P = (\psi_D \cup \psi_D^\perp) \cap \psi_S$ (hence ψ_P is interpreted as the set of all wffs of L^* whose truth value can be predicted within $\mathscr{T}$ whenever the assumptions in $\mathscr{A}$ are stated). It is apparent that ψ_P does not necessarily coincide with one of the sets ψ_T, ψ_S. We say that $\mathscr{T}$ is *t*-complete (respectively *s*-complete) with respect to a given interpretation iff a set of assumptions exists such that $\psi_T \subseteq \psi_P$ (respectively $\psi_S = \psi_P$).[5] Then *s*-completeness implies *t*-completeness; of course, the converse implication is not generally true (yet *s*-completeness and *t*-completeness coincide whenever an interpretation via correspondence rules is given, since $\psi_T \subseteq \psi_P$ implies $\psi_P = \psi_S$ in this case; note that this coincidence does not imply that $\psi_T = \psi_S$).

It is noteworthy that the above formulation of the completeness problem allows us to distinguish the "weak" EPR realism from the "strong" realism of other authors. Indeed, the former demands, from our present viewpoint, that a satisfactory theory be complete in the sense that it not only allows us to predict the truth values of all testable wffs of L^* whenever

[4]The distinction between ψ_T and ψ_S may be rather disconcerting for many physicists, since it is a widespread opinion (based on the Copenhagen interpretation of QP and on early neopositivistic positions) that only testable statements can be endowed with a truth value in QP. We think that a number of misunderstandings and seeming paradoxes occur because of this belief, and that a clear distinction must be made between the two subsets of formulas (Garola, 1992).

[5]According to a standard definition of completeness, a (noncontradictory) *theory* $\mathscr{T}$ is said to be complete with respect to a given interpretation of L^* iff a set $\mathscr{A}$ of assumptions exists such that every wff which is true according to the interpretation is a theorem of $\mathscr{T}$ (which follows from the axioms of $\mathscr{T}$ and the assumptions in $\mathscr{A}$). This definition can be shown to be equivalent to our above definition of *s*-completeness by considering the set ψ_S in place of the set of true wffs, and the set of all theorems and negations of theorems in place of the set of theorems.

suitable assumptions are given, but also the truth value of any wff which can be obtained by means of logical connectives from atomic wffs which can be tested "without disturbing the system"; therefore it requires that suitable sets of assumptions exist such that $\psi_T \subset \psi_P$, but not necessarily such that $\psi_P = \psi_S$. The latter retains that the language of a physical theory must admit an intended interpretation, or model, which corresponds to some underlying reality, so that every closed wff of L^* must be endowed with a truth value; hence a satisfactory theory must be complete in the sense that suitable sets of assumptions exist such that $\mathscr{T}$ allows us to predict the truth value of all wffs in ψ_C. Therefore strong realism requires that $\psi_P = \psi_S = \psi_C$ (usually, it is also expected that $\psi_T = \psi_C$, at least in principle). We shall see in the following that QP is incomplete in both senses; however, the epistemological constraints imposed by "weak" realism are far less restrictive than the constraints imposed by "strong" realism.

Let us come to the language L. As we have anticipated in the Introduction, we will consider L here as an "observative" sublanguage of the broader formalized language L^* that should be constructed in order to formalize completely the natural and mathematical language of physics; of course, L is endowed with the empirical interpretation specified in Section 2. In addition, we will assume henceforth that an interpretation σ of the (individual) variables is given such that, in every laboratory $i \in I$, every physical object d in the domain D_i of i is the image of some $x \in X$, so that $d = \sigma_i(x)$ (by abuse of language we call every individual variable a "physical object" in the following; see Section 2). Hence, in particular, all (atomic, molecular, quantified) wffs in L are considered here as closed wffs and are endowed with a truth value.

Let us consider the set of all testable wffs of L. We have seen in Section 2 that, whenever an interpretation σ of the variables is given, the truth value of every atomic wff of the form $S(x)$, with $S \in \mathscr{S}$, is known in every laboratory, while the truth value of every atomic formula of the form $E(x)$, with $E \in \mathscr{E}$, can be *a priori* unknown, but it can be tested; thus the atomic wffs of L either have a known truth value, so that they can be considered testable by convention, or are testable in a proper sense of the word. It follows that all wffs of L are testable in CP, where simultaneous testability of wffs of L is allowed. On the contrary, simultaneous testability can be prohibited in QP, so that some molecular formulas are not testable [a trivial example is yielded by the conjunction $Q(x) \wedge P(x)$, where Q and P are interpreted as the properties of having a definite value of the coordinate q and of its conjugate momentum p, respectively]. Thus, if we still denote by ψ_T, ψ_S, ψ (for economy of symbols) the set of all testable wffs of L, the set of all wffs of L which are endowed with a truth value, and the set of all wffs of L, respectively, we have $\psi_T = \psi_S = \psi$ in CP and $\psi_T \subset \psi_S = \psi$ in QP.

We can now consider the completeness problem with respect to L. This problem does not take the canonical form discussed above, since the axioms of CP, or QP, cannot be expressed by means of L, as we have preliminarily observed; hence we can only wonder whether in every laboratory a suitable set of specific assumptions exists from which the metalinguistic schemes of axioms of the theory allow us to deduce as theorems (negations of theorems) all true (false) wffs in $\psi_S = \psi$ ("s-completeness with respect to L"), or all true (false) wffs in ψ_T ("t-completeness with respect to L"). However, it is easy to see that the latter case implies the former in L, so that we will not distinguish between the two kinds of completeness in the following; furthermore, we agree that the expression "with respect to L" will be understood.

Our treatment of the problem will be based on the following remarks.

Remark 3.1. We recall from Section 2 that in every laboratory i the set of all states induces a partition of the domain D_i of i. Therefore we can associate to every laboratory $i \in I$ a set of specific assumptions $\mathscr{A}_i = \{S(x), S'(y), \ldots\}$ (with $x, y, \ldots \in X$, $S, S', \ldots \in \mathscr{S}$) which is such that, for every $x \in X$ and $S \in \mathscr{S}$, $S(x) \in \mathscr{A}_i$ iff $\sigma_i(x) \in \rho_i(S)$ [equivalently, $S(x)$ is true in i], so that $\mathscr{A}_i$ uniquely specifies the state of every physical object in i. Furthermore, whenever $S(x)$ is an assumption in the laboratory i, all wffs of the form $\neg S'(x)$, with $S' \in \mathscr{S}$ and $S' \neq S$, are theorems in i.

Remark 3.2. Let us consider the wff

$$A_r = (\pi_r x)(S(x) \to E(x)), \qquad \text{with} \quad r \in [0, 1], \quad x \in X, \quad S \in \mathscr{S}, \quad E \in \mathscr{E}_E$$

This formula can be interpreted (in every laboratory $i \in I$) as stating that "the physical objects in the state S have the property E with frequency r" (the arguments which allow the substitution of probabilistic statements with frequency statements in L have been discussed in G.91 and will not be recalled here). Therefore A_r exemplifies a canonical form for theorems in L following from the metalinguistic schemes of formulas expressing physical laws (we recall from G.91, Sections 1.8 and 2.6, that whenever A_r is a theorem it must be true for every laboratory $i \in \tilde{I}$ and every interpretation of the variables). Then, it follows from physical laws, both in CP and in QP, that a value r' of r exists such that $A_{r'}$ is a theorem, while A_r is the negation of a theorem whenever $r \neq r'$ (see G.91, Definition 2.6.1).

Remark 3.3. Let A_r be a theorem and let $S(x)$ be an assumption in a laboratory $i \in \tilde{I}$. Then, $E(x)$ is a theorem in i if $r = 1$; $\neg E(x)$ is a theorem in i if $r = 0$; neither $E(x)$ nor $\neg E(x)$ is a theorem in i if $0 < r < 1$. Indeed, let $r = 1$; it follows that $A_r = A_1 \equiv (\forall x)(S(x) \to E(x))$ (see G.91, Proposition 1.6.1; the symbol $\equiv$ means logical equivalence here) so that, $S(x)$ being an assumption, $E(x)$ is a theorem in i. Analogously, let $r=0$; it follows that (see G.91)

$$A_r = A_0 \equiv \neg(\exists x)(S(x) \to E(x)) \equiv (\forall x)(S(x) \to \neg E(x))$$

so that, $S(x)$ being an assumption, $\neg E(x)$ is a theorem in i. Finally, let $0<r<1$; the hypothesis that A_r is a theorem implies (see Remark 3.2) that in every laboratory $i \in \tilde{I}$ some physical objects in the state S have the property E, some do not. Then $E(x)$ cannot be a theorem in i if $S(x)$ is an assumption, since, if it were, E should be true in i for all physical objects in the state S, contrary to our above result. Analogously, $\neg E(x)$ cannot be a theorem in i.

Remark 3.4. Whenever a physical object x in a laboratory $i \in \tilde{I}$ is in a nonpure state S, there are statements of L regarding properties of x which neither are theorems nor negations of theorems in i (this occurs both in CP and in QP). Indeed, let S be a mixture of the pure states $S_1, S_2, \ldots$, with nonzero probabilities $\lambda_1, \lambda_2, \ldots$, respectively (in symbols, $S = \sum_k \lambda_k S_k$) and let E_{S_k} be the exact effect which characterizes the pure state S_k according to our definitions in Section 2. Then it follows (see G.91, Proposition 2.6.2) that the wff $(\pi_{\lambda_k} x)(S(x) \to E_{S_k}(x))$ is a theorem, with $0<\lambda_k<1$. By making use of Remark 3.3 we conclude that neither $E_{S_k}(x)$ nor $\neg E_{S_k}(x)$ can be theorems of the theory whenever $S(x)$ is an assumption, which proves our statement. However this kind of "incompleteness" can be charged to the assumptions, not to the basic laws of the theory, since nonpure states are usually retained to yield an incomplete information on physical objects.

Let us come now to the completeness problem in CP. Let us consider a laboratory $i \in \tilde{I}$ and an assumption $S(x)$, with S a pure state. Then we recall that Proposition 2.2 in Remark 2.1 holds in CP. It follows that, since $S \in \mathscr{S}_P$, the wff A_r in Remark 3.2 can be a theorem only if $r=1$ or $r=0$, hence either A_1 or A_0 is a theorem because of the same remark. By making use of Remark 3.3, we conclude that all atomic formulas of the form $E(x)$, with $E \in \mathscr{E}_E$, are either theorems or negations of theorems in i. In addition, all atomic formulas of the form $\neg S'(x)$, with $S' \in \mathscr{S}$, $S' \neq S$, are theorems of the theory in i because of Remark 3.1. We conclude that all atomic wffs of L where x appears are either theorems or negations of theorems in CP whenever x is in a pure state. Thus, bearing in mind Remark 3.4, we conclude that CP is a complete theory according to our above definition of completeness.

Let us turn to the completeness problem in QP. Because of Remark 3.4, we can restrict ourselves again to considering pure states only, but Proposition 2.2 in Remark 2.1 does not hold in QP; on the contrary, for every $S \in \mathscr{S}_p$, an effect $E \in \mathscr{E}_E$ certainly exists such that the wff $A_r = (\pi_r x)(S(x) \to E(x))$ is a theorem of the theory with $0<r<1$ (this is a consequence of the "breakdown of strict causality" that we have discussed in Section 2). It follows from Remark 3.3 that neither $E(x)$ nor $\neg E(x)$ is a theorem of the theory in a laboratory $i\in\tilde{I}$ when $S(x)$ is an assumption in i. Thus, we conclude that QP is semantically incomplete (since all wffs of L of the form A_r are either theorems or negations of theorems in QP, as we have seen in Remark 3.2, we can say that QP is incomplete with respect to "individual physical statements" while it is complete with respect to "statistical physical statements").

Our main goals in this section are thus achieved. However, our analysis can be further improved, as follows. Let us consider a state $S\in\mathcal{S}$ and let us bear in mind our definitions of $\mathscr{E}_S$, $\mathscr{E}_S^{\perp}$, and D_S in Section 2. Whenever $S(x)$ is an assumption of the theory, it can be easily proved, by making use of Remarks 3.2 and 3.3, that the wff $E(x)$ is a theorem if $E\in\mathscr{E}_S$, the negation of a theorem if $E\in\mathscr{E}_S^{\perp}$, and neither a theorem nor the negation of a theorem whenever $E\in\mathscr{E}_E\backslash D_S$. This shows that, whenever the physical object x is in the state S in a laboratory $i\in\tilde{I}$, the theory allows us to determine all properties which are "certainly true" or "certainly false" for x, but does not allow for the determination of the sets $\mathscr{E}_{ix}^{\mathrm{T}}\backslash\mathscr{E}_S$ and $\mathscr{E}_{ix}^{\mathrm{F}}\backslash\mathscr{E}_S^{\perp}$ (we say that the properties in $(\mathscr{E}_{ix}^{\mathrm{T}}\backslash\mathscr{E}_S)\cup(\mathscr{E}_{ix}^{\mathrm{F}}\backslash\mathscr{E}_S^{\perp})=\mathscr{E}_E\backslash D_S$ are "indeterminate"[6]). This result suggests that a state S can be "compatible" with a state S' in the intuitive sense that they can yield noncontradictory information; this occurs whenever every property which is certainly true (false) in S either is certainly true (false) or indeterminate in S'. Thus we introduce a binary "compatibility" relation C on the set of all states, defined as follows:

$$\text{for every}\quad S, S'\in\mathcal{S},\quad S\,C\,S' \qquad \textbf{iff} \qquad \mathscr{E}_S\cap\mathscr{E}_{S'}^{\perp}=\varnothing=\mathscr{E}_S^{\perp}\cap\mathscr{E}_{S'}$$

It is apparent that C is reflexive and symmetric, that is, it is an accessibility relation (it must be stressed that it is not necessarily transitive). We do not intend to discuss this relation in the present paper in detail; we only consider the following particular cases, which are interesting from an epistemological viewpoint.

(i) Every nonpure state $S=\sum_k \lambda_k S_k$ (see Remark 3.4) is compatible with every pure state S_k that occurs in its decomposition (indeed, $\mathscr{E}_S=\bigcap_k \mathscr{E}_{S_k}$, hence $\mathscr{E}_S^{\perp}=\bigcap_k \mathscr{E}_{S_k}^{\perp}$, so that $\mathscr{E}_S\cap\mathscr{E}_{S_k}^{\perp}=\varnothing=\mathscr{E}_S^{\perp}\cap\mathscr{E}_{S_k}$).

[6]It could be objected that QP provides information on E even if $E\in\mathscr{E}_E\backslash D_S$, since it provides the frequency r (with $0<r<1$) which makes the wff $A_r=(\pi_r x)(S(x)\rightarrow E(x))$ true (see Remark 3.2); indeed, this is the basic idea underlying some kinds of "*fuzzy logics*" for QP (see G.91), where probability values are taken as truth values in the full sense of the word. We think that the non-Tarskian truth theory underlying these logics is not adequate to formalize the concept of truth in the natural languages (hence, in the primary language of physics) from several viewpoints, and the aforesaid information on E is expressed in our framework by the metalinguistic statement that A_r is a theorem in QP.

(ii) Whenever S and S' are pure states, they are compatible iff they are not preclusive in the sense established in G.91. More explicitly, let $\#$ be the preclusion relation between pure states defined as follows:

$$\text{for every } S, S' \in \mathscr{S}_P, \quad S \# S' \quad \textbf{iff}$$
$$\text{for every } i \in \tilde{I}, \quad \rho_i(S) \cap \rho_i(E_{S'}) = \varnothing$$

Then we get

$$\text{for every } S, S' \in \mathscr{S}_P, \quad S \not{C} S' \quad \textbf{iff} \quad S \# S'$$

The proof of this statement can be easily obtained by proving first

$$\text{for every } S, S' \in \mathscr{S}_P, \quad S \not{C} S' \quad \textbf{iff} \quad E_S < E_{S'}^{\perp} \text{ and } E_{S'} < E_S^{\perp}$$

Let us comment briefly on the above results.

From a physical viewpoint, statement (i) says that any "incomplete information" on the physical objects, summarized in a nonpure state S, can be refined without introducing contradictions between the predictions of the theory (let it be CP or QP) before and after the refinement. This is a reasonable requirement, which is expected to hold in every physical theory.

More importantly, statement (ii) shows that the relation C, though introduced because of independent physical arguments, immediately reduces (on the set of pure states) to the negation of the relation $\#$, which translates a standard concept in physical theories into our framework. In particular, $\#$ coincides with the relations $\neq$ of "being different" in CP, so that the pure states S and S' are compatible iff $S = S'$ (equivalently, different pure states never can be compatible in CP); this is rather intuitive if we recall that $D_S = \mathscr{E}_E$ in CP whenever S is a pure state, so that no property of x is "indeterminate" if $S(x)$ is an assumption. Furthermore, in the Hilbert space model of QP, the pure states S and S' are preclusive (i.e., they are in the relation $\#$) iff they are represented by orthogonal vectors (see G.91, Section 2.2), so that we have that $S\,C\,S'$ iff S and S' are represented by nonorthogonal vectors. Thus we obtain a simple criterion which allows us to decide whether the information carried by two different pure states is noncontradictory in QP: a contradiction occurs iff the states are represented by vectors which are orthogonal.

4. SOME REMARKS ON MEASUREMENTS AND PARADOXES

We intend to comment briefly in the present section on some applications to measurements and paradoxes of our definitions and results in Sections 2 and 3.

(i) Let us consider a typical (pure, first-kind, ideal) quantum measurement of an observable A on a physical object x in the pure state S. Let a_1,

a_2 be possible outcomes (i.e., outcomes whose probability is not zero in the state S) and let S_1, S_2 be the corresponding pure states after the measurement. Then, $S\, C\, S_1$ and $S\, C\, S_2$ in QP, where the measurement satisfies the projection postulate, while $S_1 \# S_2$, hence $S_1 \not{C} S_2$ (see our last statement in Section 3). Physically $S\, C\, S_1$ and $S\, C\, S_2$ mean that in this kind of measurement the initial information about the physical object does not conflict with the information that we obtain by means of the measurement (so that we can attribute the obtained outcome to the object both prior and after the measurement). On the contrary, $S_1 \not{C} S_2$ means that different outcomes necessarily lead to noncompatible information. Let us show that this is not counterintuitive. Indeed, x in S_1 implies that the property E_1, interpreted as "A has value a_1," is certainly true, while x in S_2 implies that the property E_2, interpreted as "A has value a_2," is certainly true. Now, E_1 and E_2 are mutually exclusive (indeed $E_2 < E_1^{\perp}$), but E_1 and E_2 are indeterminate in the state S, so that no prediction of the truth values of $E_1(x)$ and $E_2(x)$ can be made before the measurement in QP (of course no such situation may occur in CP, since one outcome only is allowed and the state may be assumed to remain unchanged during the measurement).

(ii) Let us consider the (pure, first-kind, ideal) quantum measurements of two noncommuting observables A and B that can be performed on a physical object x in the state S (for simplicity, we assume here that S is a pure state), let a, b, respectively, be possible outcomes, and let S_a, S_b be the corresponding pure states after the measurements. Whenever S_a and S_b are represented by nonorthogonal vectors, $S_a\, C\, S_b$; since $S\, C\, S_a$ and $S\, C\, S_b$ because of (i) above, the states S, S_a, S_b express compatible information in this case.

The situation discussed in (ii) occurs in the EPR experiment, which many physicists believe exhibits some kind of paradox in QP. Here, the physical object consists of a two-particle system and the state S is a common eigenstate of two commuting observables Q and P, such that $Q = Q_\alpha - Q_\beta$ and $P = P_\alpha + P_\beta$ (where Q_α, Q_β denote the position and P_α, P_β the momentum of particles α, β, respectively; of course, $[Q_\alpha, P_\alpha] \neq 0 \neq [Q_\beta, P_\beta]$). Then, the quantum measurements which are considered consist of the measurements of Q_α and P_α. Let us denote by q and p, respectively, two possible outcomes of Q_α and P_α; by S_q, S_p the corresponding states of the whole system after the measurement; and by $|\psi_q\rangle$, $|\psi_p\rangle$ the vectors which represent these states in the Hilbert space of the system. It is well known that $|\psi_q\rangle$ can be expressed as the tensor product of the vectors $|\psi_{\alpha_q}\rangle$ and $|\psi_{\beta_q}\rangle$, which represent the states S_{α_q} and S_{β_q}, respectively, of particles α and β after the measurement of Q_α; analogously, $|\psi_p\rangle$ can be expressed as the tensor product of the vectors $|\psi_{\alpha_p}\rangle$ and $|\psi_{\beta_p}\rangle$, which represent the states S_{α_p} and S_{β_p}, respectively, of particles α and β after the measurement of P_α. It is easy to see that $0 \neq \langle\psi_q|\psi_p\rangle = \langle\psi_{\alpha_q}|\psi_{\alpha_p}\rangle\langle\psi_{\beta_q}|\psi_{\beta_p}\rangle$. Hence we get $S_q\, C\, S_p$, and also $S_{\alpha_q} C S_{\alpha_p}$ and $S_{\beta_q} C S_{\beta_p}$. This means, according to our interpretation in Section 3, that the information in the states of particles α and β after a position measurement of α does not conflict with the information in the states of particles α and β after a momentum measurement of α. Thus we conclude that the analysis of the EPR argument does not lead to counterintuitive results in our interpretation, nor to violations of basic epistemological requirements regarding physical theories (it trivially does not lead to antinomies in QP); then, using our definition of paradox in footnote 2, we conclude that no paradox occurs in the original EPR experiment, nor follows from it.

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